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EFFECT OF ELASTIC BOUNDARIES IN HYDROSTATIC PROBLEMS

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The possibility and conditions of use of the Bernoulli equation for description of an elastic pipeline were considered. It is shown that this equation is identical in form to the Bernoulli equation used for description of a rigid pipeline. It has been established that the static pressure entering into the Bernoulli equation is not identical to the pressure entering into the impulse-momentum equation. The hydrostatic problem on the pressure distribution over the height of a beaker with a rigid bottom and elastic walls, filled with a liquid, was solved.

Keywords: *impulse-momentum equation, Bernoulli equation, elastic pipeline, beaker with elastic walls.*

Introduction. The Bernoulli equation representing the first integral of the impulse-momentum equation for the ideal-liquid hydrodynamics in a uniform gravity-force field is used in calculations of a steady liquid flow. For a nonstationary liquid flow it is necessary to use the Lagrange–Cauchy integral [1]. From the physical standpoint, the Bernoulli equation represents the specific-energy balance for a steady liquid motion. It is widely used for calculating liquid flows in pipelines with changing cross sections as well as in hydrostatics problems. Therefore, it is worthwhile to analyze use of the Bernoulli equation for description of an elastic pipeline.

1. Pressure Gradient in a Cylindrical Pipeline. We first consider the basic mechanisms of a real-liquid flow in a pipeline of constant cross section. The potential energy of a liquid flowing in a constant-diameter pipeline, determined by the static pressure P_{st}, is expended to overcome the frictional force (to maintain a constant dynamic pressure P_{dyn}). Let a liquid flow from a tank through a smooth pipeline (Fig. 1). The change in the height of the liquid in the tank h is disregarded or it is assumed that this height remains constant due to the refilling of the tank with liquid. The static pressure decreases linearly from the value attained at the beginning of the pipeline, determined by the height of the liquid level in the tank $P_{st} = \rho gh$ exclusive of the dynamic pressure, to zero at the output of the pipeline. Excessive pressure is considered.

The velocity of the liquid flow in a pipeline of constant cross section and, consequently, the dynamic pressure $P_{dyn} = \rho V^2/2$ remain constant throughout the pipeline. Due to the friction (the liquid possesses a viscosity), the dynamic pressure should decrease. However, this does not take place because the initial energy determined by the static pressure P_{st} is expended to maintain a constant dynamic pressure P_{dyn}. The following energy conversion takes place in the process of flowing of the liquid in the pipeline:

$$\begin{array}{c} P_{st} \to P_{dyn} \to \text{ heat} . \\ \downarrow \\ \text{const} \end{array}$$

The pressure gradient in the pipeline $\Delta P_{st}/\Delta x$ is such that at the output of the pipeline, if it is opened, the excessive static pressure is equal to zero. The static pressure P_{st} should be completely expended during the flowing of the liquid to the opened end of the pipeline.

2. Bernoulli Equation for an Elastic Pipeline. We now consider the use of the Bernoulli equation for description of a liquid motion in an elastic pipeline. The total derivative in the impulse-momentum equation for an ideal liquid in an elastic pipeline [2] is equal to

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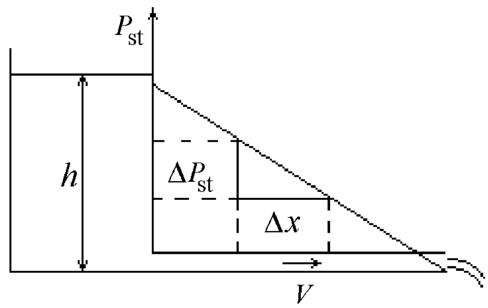

Fig. 1. Change in the static pressure P_{st} along the length of a rigid pipeline of constant cross section.

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + W\frac{\partial V}{\partial r} = -\frac{\partial\,(PS)}{\rho S\partial x}\,. \tag{1}$$

Let us rearrange Eq. (1) to the form

$$\rho\,\frac{\partial V}{\partial t} + \rho V\frac{\partial V}{\partial x} + \rho W\frac{\partial W}{\partial x} = -\frac{\partial\,(PS)}{S\partial x} = -\frac{\partial P}{\partial x} - P\,\frac{\partial S}{S\partial x} = -\frac{\partial P}{\partial x} + P\,\frac{\partial P}{D\partial x}\,. \tag{2}$$

Equation (2) was written with the use of the Hooke's law for an elastic wall

$$\partial P = -D\,\frac{\partial S}{S}\,. \tag{3}$$

The presence of the sign minus in (3) is explained by the fact that the last term in (2) defines the pressure exerted by the wall on the liquid and represents the known relation between the directions of the restoring elastic force and the displacement in the theory of vibrations. It is assumed that vortices are not formed in the liquid flow and this flow is potential [3], i.e., rot $\mathbf{V} = 0$, and, consequently, $\frac{\partial W}{\partial x} = \frac{\partial V}{\partial r}$. Moreover, the impulse-momentum equation (2) for an elastic pipeline accounts for the transverse velocity component of the flow W entering into the convective side of the total velocity derivative of the liquid.

Equation (2) can be rearranged to the form true for a rigid pipeline [2]:

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + W\frac{\partial W}{\partial x} = -\frac{\partial P_{st}}{\rho\partial x}\,, \tag{4}$$

where

$$P_{st} = P - P_w\,. \tag{5}$$

In formula (5), $P_w = P^2/(2D)$ is in fact the volume energy density of the extended pipeline wall. The quantity P_w can be identified with any pressure, the physical meaning of which will be discussed below.

In integrating Eq. (4), first of all, one should answer the question of whether a liquid flow existing in an elastic pile line is also steady or this flow can be nonstationary. Let us assume that in the case where the wall of an elastic pipeline is fairly rigid, the pipeline is subjected to a certain tensile stress, the liquid in this pipeline possesses a large viscosity, and it flows with a small velocity, the liquid flow in such an elastic pipeline can be stationary in character. It is also conceivable that under the indicated conditions, in the case of an ideal-liquid flow in an elastic pipeline, the walls of the pipeline will be deformed so slowly that the nonstationarity of the flow will be negligibly small. Therefore, integrating (4) for the stationary case, i.e., at $\partial V/\partial t = 0$, we find the Bernoulli equation for a horizontal elastic pipeline positioned at a height h = const above sea level:

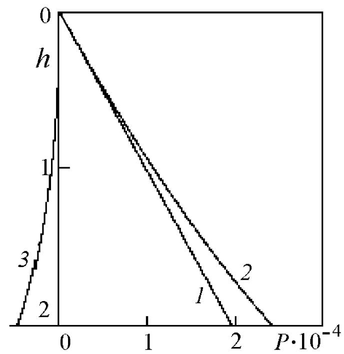

Fig. 2. Dependence of the pressures in an elastic beaker on the level of the liquid in it: 1) static pressure P_{st}; 2) total pressure $P = P_{st} + P_w$; 3) pressure on the side of the elastic wall of the beaker P_w. h, m; $P \cdot 10^{-4}$, Pa.

$$P_{st} + \frac{\rho (V^2 + W^2)}{2} = \text{const} . \tag{6}$$

For a nonhorizontal pipeline, Eq. (6) will take the form

$$P_{st} + \rho g h + \frac{\rho (V^2 + W^2)}{2} = \text{const} . \tag{7}$$

Thus, the Bernoulli equation for an elastic pipeline is identical to that for a rigid pipeline. The sum of the static, hydrostatic, and dynamic pressures in these pipelines remains constant. However, the impulse-momentum equation (1) for an elastic pipeline includes not the static pressure P_{st} but the pressure $P = P_{st} + P_w$ that is equal to the sum of the static pressure and the pressure that is caused by the reaction of the elastic wall but is not equivalent to this reaction. The elastic wall reacts to the total pressure P.

By way of example we will consider the use of the Bernoulli equation in solving the simplest problem on determination of the relation between the height h of a liquid column at rest ($V = W = 0$) in a beaker with elastic walls and a rigid bottom and the pressure at the bottom of the beaker. This problem is not trivial. The point is that the pressure representing a liquid-flow parameter is absent in the impulse-momentum equation (1) for an elastic pipeline. In fact, we introduced the pressure fairly artificially by division of the force $F = PS$ moving the liquid into the two cofactors, one of which was called the pressure P.

When the pressure is measured in reality, its value is judged indirectly or by the height of the difference between the levels of the liquid, e.g., in an U-shaped liquid column manometer, or by the deformation of any element in a metallic pressure gauge. In this case, the static pressure P_{st} is measured. It is incorrect to identify the measured pressure P_{st} with the quantity P representing the cofactor of the force F.

Using formula (7), we will find the measured static pressure

$$P_{st} = P - P_w = \rho g h . \tag{8}$$

We will also use the term pressure for the quantity P in the further discussion keeping in mind that it is not identical to the static pressure.

From the preceding it may be see that the distribution of the static pressure over the height of an elastic beaker in accordance with formula (8) is linear in character: $P_{st} = \rho g h$. Clearly this pressure distribution should be independent of the quality of the beaker walls — if they are rigid or elastic.

We now determine the distribution of all the pressures over the height of an elastic beaker (Fig. 2). Using formula (8), we write the following expression:

$$\rho g h = P - \frac{P^2}{2D} . \tag{9}$$

Solution of the quadratic equation (9) for the pressure P gives

$$P = D - \sqrt{D^2 - 2D\rho gh}\ . \tag{10}$$

The sign plus before the root is not admissible because, at $D \to \infty$, the quantity $P = \rho gh$. It should be noted that the accuracy of formulas (9) and (10) is determined by the correctness of the use of Hooke's law in the form of (3). The pressure that is due to the reaction of the elastic wall to the liquid will be determined by formula (8)

$$P_w = P - \rho gh = D - \sqrt{D^2 - 2D\rho gh} - \rho gh\ . \tag{11}$$

In Fig. 2, the straight line 1 is the static pressure P_{st}, curve 2 is the pressure P, and curve 3 is the pressure caused by the reaction of the elastic wall of the beaker to the liquid P_w. The calculation was carried out with the use of the following parameters: the elasticity of the beaker wall D = 64,000 N/m^2 and the density of the liquid ρ = 1000 kg/m^3. It is assumed that all pressures on the surface of the liquid in the beaker are equal to zero and reach a maximum value at the bottom of the beaker. In the case of a viscous-liquid flow in an elastic pipeline, a pressure balance analogous to the pressure balance presented in Par. 1 will take place:

$$\begin{array}{l} P \to P_{st} \to P_{dyn} \to \text{ heat}\ . \\ \downarrow \\ P_w \end{array}$$

The part of the pressure P, representing the static pressure P_{st}, is expended to maintain the dynamic pressure P_{dyn}, and then it is converted into heat. The other part of the pressure P, representing the pressure P_w, is expended to extend the pipeline, and then it is converted into the volume potential energy of the extended wall. The dynamic pressure P_{dyn}, unlike that in a rigid pipeline, does not remain constant — the cross section of the pipeline changes along its length because of the different extension of it in different regions. Moreover, since the extension of the pipeline is different in different regions, the part of the pressure expended for the deformation of the pipeline P_w will change when passing from section to section of the pipeline.

It should be noted that the volume energy density of the material of the walls of an absolutely rigid pipeline is equal to zero (P_w = 0). From the physical standpoint, this is explained by the fact that the walls of this pipeline are not deformed under the action of the internal pressure. An elastic body can obtain an elastic potential energy only in the case of its deformation.

Equation (4) is not informative for an elastic pipeline. It does not include the quantity P_w that is a part of the pressure $P = P_{st} + P_w$ and is responsible to a large degree for the extension of the pipeline. Equation (4), unlike (1), does not characterize the system liquid–elastic wall as a set. The static pressure P_{st} alone cannot be responsible for the extension of the pipeline and be used in Hooke's law.

The physical pattern of the relations between the different pressures in an elastic pipeline is as follows. The pressure representing a force and an energy characteristic at a time determines the volume density of the potential energy of the liquid. Let us assume that the potential energy of any volume Ω of the liquid in any region of the elastic pipeline, equal to P_{w1}/P, was expended to extend the pipeline and that downstream the extension of the pipeline is larger. Because of this, when the volume Ω moves, a larger portion of its potential-energy density $P_{w2}/P > P_{w1}/P$ is expended for the extension of the pipeline. In this case, the static pressure P_{st} can be also changed; however, in accordance with (6), this change is only due to the change in the velocity of the flow caused by the change in the area of the cross section of the pipeline.

Thus, the influence of the change in the area of the cross section of a rigid pipeline on the liquid flow in it is not identical to that of an elastic pipeline. For a rigid pipeline, a change in the velocity of the liquid flow in it influences only the first term in the equality $P = P_{st} + P_w$, and the second term is absent (the pipeline is assumed to be absolutely rigid). For an elastic pipeline, both terms change; however, the changes in these terms are due to different reasons: P_{st} changes as a result of the change in the velocity of the liquid flow caused by the geometric change in the length of the area of the cross section of the pipeline independently of the reasons caused this change, and P_w changes as a result of the extension of the elastic pipeline wall.

3. Calculation of the Form of an Elastic Beaker Filled with a Liquid. The above analysis of the use of the Bernoulli equation for an elastic pipeline allowed us to solve the problem on determination of the form of an elastic beaker with a rigid bottom filled with a liquid. However, to calculate the form of such a beaker with a liquid it is necessary to analyze Hooke's law in more detail. The simple form of the Hooke's law (3) allows one to relatively simply integrate the impulse-momentum equation (1). However, formula (3) is too rough for analysis of the elastic-beaker form.

The problem posed will be solved with the use of the Hooke's law in the form $\Delta P = -D\frac{\Delta(d\Omega)}{d\Omega}$. This form of the Hooke's law was used in [4] for investigating the vibrations in an elastic pipeline. It is assumed that the origin of coordinates $h = 0$ on the surface of the liquid in the beaker (Fig. 2). The coordinate h increases in the direction to the bottom of the beaker. Since a liquid motion is absent, the Hooke's law becomes simpler and takes the form

$$\Delta P = -D\,\frac{\Delta\,(dS)}{dS}\,. \tag{12}$$

Taking into account that $S = f(h)$, we will pass from the increments to the differentials

$$dP = -D\,\frac{\frac{d\,(dS)}{dh}\,dh}{dS} = -D\,\frac{\frac{d^2S}{dh^2}\,dh^2}{dS} = -D\,\frac{\frac{d^2S}{dh^2}}{\frac{dS}{dh}}\,dh\,. \tag{13}$$

Let us rearrange formula (13):

$$\frac{dP}{dh} = -D\,\frac{d}{dh}\left[\ln\left(\frac{dS}{dh}\right)\right]. \tag{14}$$

Equation (14) can be integrated once:

$$\frac{P}{D} = -\ln\frac{1}{C_1}\left(\frac{dS}{dh}\right). \tag{15}$$

Consequently

$$\frac{dS}{dh} = C_1\exp\left(-\frac{P}{D}\right). \tag{16}$$

We will use the relation between the pressure P and the height h in the form of (9). It should, be noted that formula (9) was derived from the simpler form of the Hooke's law (3) as compared to (12). Therefore, at this stage of the analysis an additional approximation is used because the impulse-momentum equation (1) is difficult to analytically solve with the use of the Hooke's law in the form of (12). Because of this, the result of this solution — the function $P_w = f(P)$ analogous to the function $P_w = P^2/2D$ determined in Par. 2 with the use of the Hooke's law in the form of (3) — cannot be used.

Differentiation of (9) gives

$$\frac{dh}{dP} = \frac{1}{\rho g}\left(1 - \frac{P}{D}\right). \tag{17}$$

Multiplying (16) by (17), we obtain

$$\frac{dS}{dP} = \frac{C_1}{\rho g}\exp\left(-\frac{P}{D}\right)\left(1 - \frac{P}{D}\right). \tag{18}$$

Integration of (18) gives

$$S = \frac{C_1}{\rho g} P \exp\left(-\frac{P}{D}\right) + C_2 . \tag{19}$$

Taking into account the fact that the pressure on the surface of the liquid $P = 0$ (Fig. 2), we find $C_2 = S_0$ and write formula (19) in the form

$$S - S_0 = \frac{C_1}{\rho g} P \exp\left(-\frac{P}{D}\right). \tag{20}$$

Moreover, since $S = S_H$,

$$S_H - S_0 = \frac{C_1}{\rho g} P_H \exp\left(-\frac{P_H}{D}\right). \tag{21}$$

Dividing (20) by (21), we obtain the relation between the cross-sectional area of the elastic beaker and the pressure on it

$$\frac{S - S_0}{S_H - S_0} = \exp\left[-\left(\frac{P}{D} - \frac{P_H}{D}\right)\right] \frac{P}{P_H} . \tag{22}$$

It is difficult to judge the area of the liquid on the surface of the beaker S_0 in advance because this area is determined by the bending properties of the beaker walls.

Let us analyze the dependence $S = f(P)$ obtained. We will determine the pressure at which the cross-sectional area of the beaker will be maximum. Taking a derivative of (22) and equating it to zero, we obtain the maximum cross-sectional area of the beaker S_m attained at $P_m = D$. It follows from formula (22) that

$$\frac{S_m - S_0}{S_H - S_0} = \exp\left[-\left(1 - \frac{P_H}{P_m}\right)\right] \frac{P_m}{P_H} . \tag{23}$$

Formula (23) defines the bending properties of the beaker walls because it allows one to calculate the relation between the areas S_m, S_0, and S_H.

In accordance with (9), for a fairly rigid wall, we will use the simplest dependence $P = f(h)$ to ensure that the complexity of the formulas being used was not increased at the final stage of the analysis. We will assume that the pressure P is proportional to the height of the liquid h measured from its surface to the bottom of the beaker, $P \approx \rho g h$. In fact, we equated the pressure P to the static pressure P_{st}.

By way of example, we will assume that $\frac{P_H}{P_m} \approx \frac{H}{h_m} = \frac{3}{2}$, where h_m is the position of the cross section with a maximum area S_m of the elastic beaker. Thus, it is assumed that the properties of the beaker wall are such that the area S_m is realized at a height 1/3 from the bottom of the beaker. Consequently, according to (23),

$$\frac{S_m - S_0}{S_H - S_0} = \frac{P_m}{P_H} \exp\left[-\left(1 - \frac{P_H}{P_m}\right)\right] \approx 1.1 . \tag{24}$$

Since, in this case, $P_H = \frac{3}{2} P_m = \frac{3}{2} D = \rho g H$, the relation between the height of the liquid in the beaker and the elasticity of its wall will be as follows: $H = 3D/(2\rho g)$. In practical calculations, the order of analysis is different. The position h_m of the maximum cross-sectional area of the beaker is determined by the height of the liquid in it H and the elasticity of its wall D. Moreover, it is unlikely that the relation $P \approx \rho g h$ can be used in practical calculations.

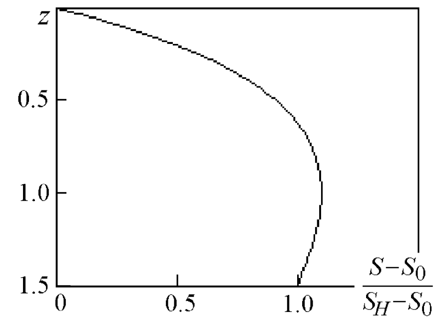

Fig. 3. Dependence of the relative cross-sectional area of the elastic beaker with a liquid $(S - S_0)/(S_H - S_0)$ on the dimensionless coordinate z along the height of the beaker.

It seems likely that the dependence $P = f(h)$ can be defined by the approximate formula (10); however, in this case, the calculations become much more complex.

Taking into account the fact that $P_m = D$, we will write formula (22) in the form

$$\frac{S - S_0}{S_H - S_0} = \frac{2}{3}\frac{P}{D}\exp\left[-\left(\frac{P}{D} - \frac{3}{2}\right)\right]. \tag{25}$$

Figure 3 shows the dependence of the relative area of the section of the elastic beaker on the value of $z = P/D \approx \rho gh/D$ determined by formula (25).

Conclusions. Our analysis has shown that the Bernoulli equation for an elastic pipe line is identical to that for a rigid pipeline. The problem on determination of the shape of an elastic beaker with a rigid bottom, considered by us as an example, is the simplest problem illustrating its character. However, this problem reflects all features of the hydrostatic relation between the pressure in an elastic pipeline and its wall.

NOTATION

C_1, integration constant, m; C_2, integration constant, m^2; D, elasticity of the wall of a pipeline, N/m^2; F, force moving the liquid in the pipeline, N; g, free fall acceleration, m/sec^2; H, height of the liquid in an elastic beaker, m; h, current height of the liquid in the elastic beaker, m; P, pressure, Pa; P_H, pressure at the bottom of the beaker, Pa; P_{st}, static pressure, Pa; P_{dyn}, dynamic pressure, Pa; P_w, P_{w1}, P_{w2}, current pressure (equivalent to the volume density of the energy of the extended wall of the pipeline) at cross sections 1 and 2, Pa; P_m, pressure at the site of maximum extension of the beaker, Pa; r, radial coordinate, m; S, area of the cross section of a thin-wall pipeline, m^2; S_0, area of the liquid surface in the beaker, m^2; S_H, area of the rigid bottom of the beaker, m^2; S_m, maximum cross section of the beaker, m^2; t, time, sec; $\mathbf{V}$, velocity vector of the liquid in the pipeline, m/sec; V, W, longitudinal and transverse velocity components of the liquid in the pipeline, m/sec; x, longitudinal coordinate, m; z, dimensionless coordinate along the height of the beaker; ρ, density of the liquid in the beaker, kg/m^3; Ω, volume of the liquid in the beaker, m^3; $d\Omega$, differential volume of the liquid in the beaker, m^3. Subscripts: H, height; m, maximum; dyn, dynamic; w, wall; st, static.

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